A GENERALIZATION OF LEVI-CIVITA'S PARALLELISM AND THE FRENET FORMULAS*

BY

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Introduction. In the Riemann geometry of n dimensions the arc length t of a curve is given by the value of an integral of the form

$$(1) \qquad t = \int_{u_1}^{u} \sqrt{g_{\alpha\beta}\, x'^{\alpha}\, x'^{\beta}}\, du,$$

where the coefficients $g_{\alpha\beta}$ are functions of $x^1, \cdots, x^n$ only, and where the notation implies that α and β are summed from 1 to n. The vector analysis of such a geometry has been rather systematically developed.† Levi-Civita has developed a theory of parallelism‡ for the Riemann space, in which a vector ξ defined at each point of a curve is said to remain parallel to itself as it moves along the curve if it satisfies the system of equations

$$(2) \qquad \frac{d\xi^{\lambda}}{du} + \{\alpha\beta, \lambda\}\, x'^{\beta}\, \xi^{\alpha} = 0 \qquad (\lambda = 1, \cdots, n),$$

where $\{\alpha\beta, \lambda\}$ is the Christoffel symbol of the second kind formed with respect to the coefficients $g_{\alpha\beta}$ occurring in the expression (1) for the arc length. He has shown that if ξ_1 and ξ_2 are two variable vectors which are defined for each point of a curve, and which satisfy equations (2), then the angle between the vectors remains constant as they move along the curve.

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This paper is essentially as submitted for a thesis to The University of Chicago, and was prepared with the cooperation of Professor G. A. Bliss.

† Some of the more recent works treating of this subject are

D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie*, 1923, which also contains an excellent bibliography;

F. D. Murnaghan, *Vector Analysis and the Theory of Relativity*, 1922;

A. S. Eddington, *The Mathematical Theory of Relativity*, 1923;

G. Ricci et T. Levi-Civita, *Méthodes de calcul différentiel absolu et leurs applications*, Mathematische Annalen, vol. 54 (1901), pp. 125–201.

‡ T. Levi-Civita, *Nozione di parallelismo in una varietà qualunque et conseguente specificazione geometrica della curvatura Riemanniana*, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173-205.

In the present paper a more general space is considered in which the arc length is defined by the value of an integral of the form

$$t = \int_{u_1}^{u} F(x^1, \cdots, x^n; x'^1, \cdots, x'^n)\, du. \tag{3}$$

The geometry of such a space has been considered by Finsler.*

The measurement of vectors and angles in a Riemann space is with respect to the matrix of coefficients $g_{\alpha\beta}$. In the more general space here considered the matrix used for this purpose is

$$f_{\alpha\beta} = \frac{\partial F}{\partial x'^\alpha}\frac{\partial F}{\partial x'^\beta} + F\frac{\partial^2 F}{\partial x'^\alpha \partial x'^\beta},$$

and hence in general the angle between two vectors depends upon a parameter direction, as well as upon the point in space at which the vectors are taken. An interesting geometric interpretation is given to this situation in terms of the "indicatrix" of the calculus of variations associated with the integral (3), and a quadratic manifold which osculates the indicatrix.

In the present paper a differentiation operation corresponding to the left member of (2) is developed, and by means of it many of the results of Levi-Civita are generalized for the more general space here under consideration.

Blaschke† by means of the parallelism of Levi-Civita has derived the Frenet formulas for a twisted curve in a Riemann space of n dimensions. By following a method analogous to that of Blaschke, and making use of the extensions of the notions of parallelism and the measurement of angles which are here developed, it is found that the Frenet formulas may be obtained for any space in which the arc length is given by (3).

1. **The functions F and f.** Let the equations of a curve C in an n-dimensional space be

$$x^\alpha = x^\alpha(u), \quad u_1 \leq u \leq u_2 \qquad (\alpha = 1, \cdots, n).$$

We suppose the space to be such that the arc length t of the curve is given by the value of an integral of the form

$$t = \int_{u_1}^{u} F(x, x')\, du, \tag{3}$$

* P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen,* Dissertation, 1918.

† W. Blaschke, *Frenets Formeln für den Raum von Riemann,* Mathematische Zeitschrift, vol. 6 (1920), pp. 94–99.

where we have employed the vector notation x and x' to represent the vectors $(x^1, \cdots, x^n)$ and $(x'^1, \cdots, x'^n)$ respectively, and where the primes denote derivatives with respect to u. A necessary and sufficient condition that the value of the integral (3) shall be independent of the parametric representation of the curve along which the integral is taken is that F shall satisfy the homogeneity condition*

$$F(x, \varkappa x') = \varkappa F(x, x'), \qquad \varkappa > 0. \tag{4}$$

We shall suppose the function F to be positive and to satisfy the condition (4) at every point of the region of space which we are considering. Furthermore it will be assumed that the quadratic form

$$F_{\alpha\beta}\,\xi^\alpha \xi^\beta > 0, \qquad \xi \neq x',$$

where the symbol $F_{\alpha\beta}$ defined by $F_{\alpha\beta} = \partial^2 F/\partial x'^\alpha \partial x'^\beta$ has been introduced for convenience of notation. Here α and β are summed from 1 to n, as will be always understood in the following whenever an index letter occurs twice in the same term. The curve C and the function F will be considered to be real and to possess such continuity properties as may be required in the subsequent development.

As a consequence of the homogeneity condition (4) the relations

$$F_\alpha x'^\alpha = F, \qquad F_{\alpha\beta}\, x'^\alpha = 0 \tag{5}$$

are identically satisfied.

Let a function $f(x, x')$ be introduced by the equation

$$F = \sqrt{2f}$$

where now $f = F^2/2$ satisfies the homogeneity condition

$$f(x, \varkappa x') = \varkappa^2 f(x, x'), \qquad \varkappa > 0,$$

and the resulting identities obtained by differentiation,

$$\begin{aligned} & f_\alpha x'^\alpha = 2f, \qquad f_{\alpha\beta}\, x'^\alpha = f, \\ & f_{\alpha\beta\gamma}\, x'^\alpha = 0, \qquad f_{\alpha\beta} = F_\alpha F_\beta + F F_{\alpha\beta} = f_{\beta\alpha}. \end{aligned} \tag{6}$$

Here the subscripts applied to f indicate the partial derivatives with respect to the corresponding x' variables. We assume the determinant $f_{\alpha\beta}$ to be

* O. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 195.

different from zero and denote by $f^{\alpha\beta}$ the element of the reciprocal matrix corresponding to $f_{\alpha\beta}$. Then

$$f_{\alpha\beta} f^{\alpha\varepsilon} = \delta^{\varepsilon}_{\beta}, \tag{7}$$

where $\delta^{\beta}_{\beta} = 1$ and $\delta^{\varepsilon}_{\beta} = 0$ for $\beta \neq \varepsilon$. This assumption that the determinant $f_{\alpha\beta} \neq 0$ may be shown to be equivalent to supposing the F_1 function of the calculus of variations to be non-vanishing.*

Consider the "indicatrix" of the calculus of variations† associated with the integral (3), and which is defined to be the manifold in the space of the x' variables determined by the equation $F(x, x') = 1$, where we regard x as fixed. On account of (5) and (6) the equation of the indicatrix may be given the form

$$\sqrt{f_{\alpha\beta}(x, x')\, x'^{\alpha} x'^{\beta}} = 1.$$

The equation

$$\sqrt{f_{\alpha\beta}(x, r)\, x'^{\alpha} x'^{\beta}} = .$$

for a fixed x and r defines another manifold in the x' space, which may well be called the *osculating indicatrix,* since it has contact of the second order with the original indicatrix at an arbitrary point $x' = r$ on it.‡ In the Riemann geometry these two manifolds are clearly coincident.

2. **Tensors and invariants.** Let the equations

$$\begin{aligned} y^i &= y^i(x^1, \cdots, x^n), & y'^i &= \frac{\partial y^i}{\partial x^\alpha} x'^\alpha, \\ x^\alpha &= x^\alpha(y^1, \cdots, y^n), & x'^\alpha &= \frac{\partial x^\alpha}{\partial y^i} y'^i, \end{aligned} \tag{8}$$

where the primes denote derivatives with respect to u, define a regular extended point transformation T and its inverse. We record here for future reference some relations obtained from (8):

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* For a definition of the F_1 function and some of its properties, see M. Mason and G. A. Bliss, *The properties of curves in space which minimize a definite integral,* these Transactions, vol. 9 (1908), p. 441.

For a proof of the statement made above, see J. H. Taylor, *Reduction of Euler's equations to a canonical form,* Bulletin of the American Mathematical Society, vol. 31 (1925).

† C. Carathéodory, *Über die starken Maxima und Minima bei einfachen Integralen,* Mathematische Annalen, vol. 62 (1906), p. 456; also Bolza, loc. cit., p. 247.

‡ P. Finsler, loc. cit., p. 42.

$$
(9) \qquad
\begin{aligned}
y''^i &= \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta} x'^\alpha x'^\beta + \frac{\partial y^i}{\partial x^\alpha} x''^\alpha, \\
x''^\alpha &= \frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} y'^i y'^k + \frac{\partial x^\alpha}{\partial y^i} y''^i;
\end{aligned}
$$

$$
(10) \qquad \frac{\partial y'^i}{\partial x^\beta} = \frac{\partial^2 y^i}{\partial x^\beta \partial x^\alpha} x'^\alpha, \qquad \frac{\partial x'^\alpha}{\partial y^k} = \frac{\partial^2 x^\alpha}{\partial y^k \partial y^i} y'^i;
$$

$$
(11) \qquad \frac{\partial y'^i}{\partial x'^\beta} = \frac{\partial y^i}{\partial x^\beta}, \qquad \frac{\partial x'^\alpha}{\partial y'^k} = \frac{\partial x^\alpha}{\partial y^k}.
$$

In most of the literature dealing with the tensor analysis the tensor components or coefficients are considered to be point functions. In the present paper, however, the tensor components will be allowed to be functions of $x^1, \cdots, x^n$ and their derivatives with respect to a scalar variable u.* With this extension in mind we adopt the formal definitions of tensors which are given in the books dealing with the subject.† Any set of n quantities $X^\alpha(x, x', x'', \cdots)$ $(\alpha = 1, \cdots, n)$ which transform by the extended transformation T into n new quantities $Y^i(y, y', y'', \cdots)$ in such a way that

$$
Y^i = X^\alpha \frac{\partial y^i}{\partial x^\alpha}
$$

will be called a *contravariant tensor of rank* 1. Of course the substitution from one system of coördinates to the other must be completely carried out by adjoining to (8) and (9) corresponding relations involving higher derivatives if necessary. A *covariant tensor of rank* 1 is a set of n quantities X_α which transform by T into

$$
Y_i = X_\alpha \frac{\partial x^\alpha}{\partial y^i}.
$$

If a set of n^3 quantities $X^\alpha{}_{\beta\gamma}$ transform by T into

$$
Y^i_{jk} = X^\alpha_{\beta\gamma} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k},
$$

they are said to constitute a mixed tensor, contravariant of rank 1 and covariant of rank 2. The extension of the definitions to tensors of any rank is immediate.

* For some examples of tensors of this kind see F. D. Murnaghan, loc. cit., p. 88.

† D. J. Struik, loc. cit., p. 17; F. D. Murnaghan, loc. cit., p. 17; A. S. Eddington, loc. cit., pp. 51–52.

Denote by $h(y, y')$ the result of transforming $f(x, x')$. Then

$$h(y, y') = f(x, x')$$

and hence

$$\frac{\partial h}{\partial y'^i} = \frac{\partial f}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial y'^i} = \frac{\partial f}{\partial x'^\alpha}\frac{\partial x^\alpha}{\partial y^i}$$

by (11), that is, $\partial f/\partial x'^\alpha$ is a covariant tensor of rank 1.* A second differentiation gives

$$\frac{\partial^2 h}{\partial y'^i \partial y'^j} = \frac{\partial^2 f}{\partial x'^\alpha \partial x'^\beta}\frac{\partial x'^\beta}{\partial y'^j}\frac{\partial x^\alpha}{\partial y^i}$$

$$= \frac{\partial^2 f}{\partial x'^\alpha \partial x'^\beta}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\alpha}{\partial y^i}.$$

Therefore, $f_{\alpha\beta} = \partial^2 f/\partial x'^\alpha \partial x'^\beta$ is a covariant tensor of rank 2. Similarly $f_{\alpha\beta\gamma} = \partial f_{\alpha\beta}/\partial x'^\gamma$ is a covariant tensor of rank 3. From the fact that $f_{\alpha\beta}$ is a covariant tensor of rank 2 it follows that $f^{\alpha\beta}$ is a contravariant tensor of rank 2.†

We define the *inner product* of two contravariant vectors (tensors of rank 1) ξ_1 and ξ_2 to be the value of the bilinear form

$$f_{\alpha\beta}(x, r)\xi_1^\alpha \xi_2^\beta = (\xi_1, \xi_2)_r = (\xi_2, \xi_1)_r.$$

The value of this form is an *invariant* under the transformation T.‡

Note that in general the inner product of two vectors depends upon a parameter direction vector r; in case it does not it can be shown that the space is Riemannian.§ It will be observed, however, that the value of $(\xi_1, \xi_2)_r$ is independent of the magnitude of r since $f_{\alpha\beta}$ is homogeneous of degree zero in x'. If the directions ξ_1 and ξ_2 are such that $(\xi_1, \xi_2)_r = 0$, they will be said to be *orthogonal-r*. The value of the quadratic form

$$(\xi, \xi)_r = f_{\alpha\beta}\xi^\alpha \xi^\beta$$

will be taken as the *r-norm* of ξ. To justify this terminology it will be desirable to show that the value of the form $(\xi, \xi)_r$ is always positive

* F. D. Murnaghan, loc. cit., p. 88.

† F. D. Murnaghan, loc. cit., p. 42.

‡ F. D. Murnaghan, loc. cit., p. 40; A. S. Eddington, loc. cit., p. 53.

§ See P. Finsler, loc. cit., p. 40.

and greater than zero unless $\xi = (0, \cdots, 0)$, in which case it is obviously zero. Expressing $f_{\alpha\beta}$ in terms of F we have

$$(\xi, \xi)_r = (F_\alpha \xi^\alpha)^2 + F F_{\alpha\beta} \xi^\alpha \xi^\beta,$$

the arguments in F being x and r. Under the hypotheses of § 1 these two terms cannot vanish simultaneously. For, the first term is zero only when ξ is *transversal* to r.* Moreover it follows from the condition (5) that a direction transversal to r is different from r. The second term in the expression for $(\xi, \xi)_r$ vanishes only for $\xi = r$, in which case $(F_\alpha \xi^\alpha)^2$ becomes $F^2 > 0$ by (5). A vector whose norm is 1 will be said to be *unitary*.

It will be noticed that these invariants as here defined are not associated directly with the indicatrix as are the corresponding ones in the case of the Riemann geometry; they do, however, bear a similar relation to the osculating indicatrix. In the case of two orthogonal-r directions the situation may be characterized as follows: If ξ_1 and ξ_2 are orthogonal-r, they are conjugate directions in the sense of analytic geometry, not with respect to the indicatrix, but with respect to an osculating indicatrix which is uniquely determined by r.

3. **Generalization of Levi-Civita's parallelism.** Let X be a contravariant tensor of rank 1,

$$Y^m = X^\alpha \frac{\partial y^m}{\partial x^\alpha}.$$

Differentiating with respect to u we obtain

$$\frac{d Y^m}{du} = \frac{d X^\alpha}{du} \frac{\partial y^m}{\partial x^\alpha} + X^\alpha \frac{\partial^2 y^m}{\partial x^\alpha \partial x^\beta} x'^\beta. \tag{12}$$

In the case of the Riemann geometry, the elimination of $\partial^2 y^m / \partial x^\alpha \partial x^\beta$ from the last term by means of the Christoffel transformation equations† leads to a contravariantive expression,

$$\frac{dX^\alpha}{du} + \{\lambda\beta, \alpha\} x'^\beta X^\lambda, \tag{13}$$

which forms the basis of many of the results of the paper by Levi-Civita on parallelism,‡ and is the differentiation process used by Blaschke in obtaining the Frenet formulas for a Riemann space.§ In the more general

* O. Bolza, loc. cit., p. 303; P. Finsler, loc. cit., p. 36.

† A. S. Eddington, loc. cit., p. 66; F. D. Murnaghan, loc. cit., p. 92.

‡ T. Levi-Civita, loc. cit.

§ W. Blaschke, loc. cit.

space under consideration here it is possible to make a desirable substitution for the coefficient of X^α in (12) which will give a generalization of the expression (13) having many of its properties.

The Christoffel three-index symbols are defined as follows:*

$$(14)\qquad [\alpha\beta,\lambda] = \frac{1}{2}\left(\frac{\partial f_{\alpha\lambda}}{\partial x^\beta} + \frac{\partial f_{\beta\lambda}}{\partial x^\alpha} - \frac{\partial f_{\alpha\beta}}{\partial x^\lambda}\right) = [\beta\alpha,\lambda],$$

$$(15)\qquad \{\alpha\beta,\lambda\} = \Gamma_{\alpha\beta}{}^{\lambda} = f^{\lambda\mu}[\alpha\beta,\mu] = \Gamma_{\beta\alpha}{}^{\lambda}.$$

They satisfy the relations

$$(16)\qquad f_{\lambda\mu}\Gamma_{\alpha\beta}{}^{\lambda} = [\alpha\beta,\mu]$$

and

$$(17)\qquad [\alpha\beta,\lambda]+[\lambda\beta,\alpha] = \frac{\partial f_{\alpha\lambda}}{\partial x^\beta}.$$

In the present instance these symbols are functions of x and x', as they are formed from the $f_{\alpha\beta}(x,x')$. In taking the partial derivatives with respect to x the x' variables are treated as constants.

We have seen that $f_{\alpha\beta}$ is a covariant tensor of rank 2, i. e.,

$$h_{ik} = f_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}.$$

Let us differentiate this identity with respect to y^j, remembering that the right member is not only a function of y through x but also through x':

$$(18_1)\qquad \begin{aligned}\frac{\partial h_{ik}}{\partial y^j} &= f_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial y^i\partial y^j}\frac{\partial y^\beta}{\partial x^k} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial^2 x^\beta}{\partial y^k\partial y^j}\right)\\ &\quad + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x^\gamma}{\partial y^j}\frac{\partial f_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^j}f_{\alpha\beta\gamma}.\end{aligned}$$

Now form the two similar expressions

$$(18_2)\qquad \begin{aligned}\frac{\partial h_{jk}}{\partial y^i} &= f_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial y^j\partial y^i}\frac{\partial x^\beta}{\partial y^k} + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial^2 x^\beta}{\partial y^k\partial y^i}\right)\\ &\quad + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x^\gamma}{\partial y^i}\frac{\partial f_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^i}f_{\alpha\beta\gamma},\end{aligned}$$

* L. Bianchi, *Lezioni de Geometria Differenziale*, 1902, vol. 1, pp. 64–65; F. D. Murnaghan, loc. cit., p. 89.

$$
(18_3)\qquad \frac{\partial h_{ij}}{\partial y_k} = f_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^k}\frac{\partial x^\beta}{\partial y^j} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial^2 x^\beta}{\partial y^j \partial y^k}\right) + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\frac{\partial f_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x'^\gamma}{\partial y^k} f_{\alpha\beta\gamma}.
$$

One half the sum of the first two of these expressions minus the third is the Christoffel index symbol $[ij, k]^*$ for the y-coördinate system, and we find

$$
(19)\qquad [ij,k]^* = f_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\frac{\partial x^\beta}{\partial y^k} + [\alpha\beta,\gamma]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k} + \frac{1}{2}\left(\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^j} f_{\alpha\beta\gamma} + \frac{\partial x^\alpha}{\partial y^j}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^i} f_{\alpha\beta\gamma} - \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x'^\gamma}{\partial y^k} f_{\alpha\beta\gamma}\right).
$$

Multiply (19) by $h^{km}(\partial x^\lambda/\partial y^m)\,y'^j$ and obtain

$$
(20)\qquad \begin{aligned}
\Gamma^{*m}_{ij}\frac{\partial x^\lambda}{\partial y^m}y'^j = {} & f_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\frac{\partial x^\beta}{\partial y^k} h^{km}\frac{\partial x^\lambda}{\partial y^m}y'^j \\
& + [\alpha\beta,\gamma]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k} h^{km}\frac{\partial x^\lambda}{\partial y^m}y'^j \\
& + \frac{1}{2}\frac{\partial x^\alpha}{\partial y^i} f_{\alpha\beta\gamma}\frac{\partial x^\beta}{\partial y^k} h^{km}\frac{\partial x^\lambda}{\partial y^m}\frac{\partial x'^\gamma}{\partial y^j}y'^j \\
& + \frac{1}{2}\frac{\partial x'^\gamma}{\partial y^i}\frac{\partial x^\beta}{\partial y^k} h^{km}\frac{\partial x^\lambda}{\partial y^m} f_{\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial y^j}y'^j \\
& - \frac{1}{2}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x'^\gamma}{\partial y^k} h^{km}\frac{\partial x^\lambda}{\partial y^m} f_{\alpha\beta\gamma}\frac{\partial x^\beta}{\partial y^j}y'^j,
\end{aligned}
$$

where the terms have been grouped to facilitate the next reduction. The last two terms of the right hand expression reduce to zero by the third of (6). Remembering the contravariant properties of h^{km}, and substituting for $\partial x'^\gamma/\partial y^j$ from (10), it is seen that the above equation reduces to

$$
\Gamma^{*m}_{ij}\frac{\partial x^\lambda}{\partial y^m}y'^j = f_{\alpha\beta} f^{\beta\lambda}\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}y'^j + [\alpha\beta,\gamma] f^{\gamma\lambda}\frac{\partial x^\alpha}{\partial y^i}x'^\beta + \frac{1}{2}\frac{\partial x^\alpha}{\partial y^i} f_{\alpha\beta\gamma}\frac{\partial x^\beta}{\partial y^k} h^{km}\frac{\partial x^\lambda}{\partial y^m}\frac{\partial^2 x^\gamma}{\partial y^i \partial y^r}y'^r y'^j.
$$

By (7), (15) and (9) this becomes

$$\Gamma_{ij}^{*m} \frac{\partial x^\lambda}{\partial y^m} y'^j = \frac{\partial^2 x^\lambda}{\partial y^i \partial y^j} y'^j + \Gamma_{\alpha\beta}{}^\lambda \frac{\partial x^\alpha}{\partial y^i} x'^\beta$$
$$+ \frac{1}{2} \frac{\partial x^\alpha}{\partial y^i} f_{\alpha\beta\gamma} \frac{\partial x^\beta}{\partial y^k} h^{km} \frac{\partial x^\lambda}{\partial y^m} \left(x''^\gamma - \frac{\partial x^\gamma}{\partial y^j} y''^j \right),$$

and hence

$$(21) \quad \left(\Gamma_{ij}^{*m} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm} \right) \frac{\partial x^\lambda}{\partial y^m}$$
$$= \frac{\partial^2 x^\lambda}{\partial y^i \partial y^j} y'^j + \left(\Gamma_{\alpha\beta}{}^\lambda x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda} \right) \frac{\partial x^\alpha}{\partial y^i},$$

where the covariant property of $f_{\alpha\beta\gamma}$ has been made use of, and where the quantities in the parentheses have been made symmetric by an interchange of j and k in the second term of the left hand expression. Interchanging x and y with a corresponding change in the index letters gives

$$(22) \quad \left(\Gamma_{\alpha\beta}{}^\lambda x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda} \right) \frac{\partial y^m}{\partial x^\lambda}$$
$$= \frac{\partial^2 y^m}{\partial x^\alpha \partial x^\beta} x'^\beta + \left(\Gamma_{ij}^{*m} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm} \right) \frac{\partial y^i}{\partial x^\alpha}.$$

By means of this relation (12) becomes

$$\frac{dY^m}{du} + \left(\Gamma_{ij}^{*m} y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm} \right) Y^i$$
$$= \left[\frac{dX^\alpha}{du} + \left(\Gamma_{\lambda\beta}{}^\alpha x'^\beta + \frac{1}{2} x''^\gamma f_{\lambda\beta\gamma} f^{\beta\alpha} \right) X^\lambda \right] \frac{\partial y^m}{\partial x^\alpha}.$$

Hence we have the conclusion

If X^α is a contravariant tensor of rank 1, then θX^α is also a contravariant tensor of rank 1, the θ-process being defined by the equation

$$(23) \qquad \theta X^\alpha = \frac{dX^\alpha}{du} + \left(\Gamma_{\lambda\beta}{}^\alpha x'^\beta + \frac{1}{2} x''^\gamma f_{\lambda\beta\gamma} f^{\beta\alpha} \right) X^\lambda.$$

An analogous differentiation process may readily be established for a covariant vector. For, let X_α be a covariant tensor of rank 1. Then

$$Y_i = X_\alpha \frac{\partial x^\alpha}{\partial y^i},$$

from which

$$\frac{dY_i}{du} = \frac{dX_\alpha}{du}\frac{\partial x^\alpha}{\partial y^i} + X_\alpha \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j.$$

Substituting from (21) this reduces to

$$\frac{dY_i}{du} - \left(\Gamma^{*\,m}_{ij}\, y'^j + \frac{1}{2} y''^k h_{ijk} h^{jm}\right) Y_m$$
$$= \left[\frac{dX_\alpha}{du} - \left(\Gamma_{\alpha\beta}^{\ \ \lambda} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda}\right) X_\lambda\right]\frac{\partial x^\alpha}{\partial y^i}.$$

We therefore have the theorem

If X_α is a covariant tensor of rank 1, then θX_α is also a covariant tensor of rank 1, the θ-process in this case being defined by

$$\theta X_\alpha = \frac{dX_\alpha}{du} - \left(\Gamma_{\alpha\beta}^{\ \ \lambda} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\lambda}\right) X_\lambda. \tag{24}$$

Let us consider one more special case, that of a covariant tensor of rank 2,

$$Y_{ij} = X_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}.$$

By differentiation we obtain

$$\frac{dY_{ij}}{du} = \frac{dX_{\mu\nu}}{du}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j} + X_{\mu\nu}\frac{\partial^2 x^\mu}{\partial y^i \partial y^k} y'^k \frac{\partial x^\nu}{\partial y^j} + X_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\frac{\partial^2 x^\nu}{\partial y^j \partial y^k} y'^k.$$

By means of (21) this equation becomes

$$\frac{dY_{ij}}{du} = \frac{dX_{\mu\nu}}{du}\frac{\partial x^\mu}{\partial y^i}\frac{\partial x^\nu}{\partial y^j}$$
$$+ X_{\mu\nu}\left[\left(\Gamma^{*\,m}_{ik}\, y'^k + \frac{1}{2} y''^r h_{ikr} h^{km}\right)\frac{\partial x^\mu}{\partial y^m}\right.$$
$$\left. - \left(\Gamma_{\alpha\beta}^{\ \ \mu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right)\frac{\partial x^\alpha}{\partial y^i}\right]\frac{\partial x^\nu}{\partial y^j}$$
$$+ X_{\mu\nu}\frac{\partial x^\mu}{\partial y^i}\left[\left(\Gamma^{*\,m}_{jk}\, y'^k + \frac{1}{2} y''^r h_{jkr} h^{km}\right)\frac{\partial x^\nu}{\partial y^m}\right.$$
$$\left. - \left(\Gamma_{\alpha\beta}^{\ \ \nu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right)\frac{\partial x^\alpha}{\partial y^j}\right],$$

and hence

$$\frac{dY_{ij}}{du} - \left(\Gamma^{*\,m}_{ik}\, y'^k + \frac{1}{2}\, y''^r\, h_{ikr}\, h^{km}\right) Y_{mj} - \left(\Gamma^{*\,m}_{jk}\, y'^k + \frac{1}{2}\, y''^r\, h_{jkr}\, h^{km}\right) Y_{im}$$
$$= \left[\frac{dX_{\mu\nu}}{du} - \left(\Gamma_{\mu\beta}{}^{\alpha}\, x'^{\beta} + \frac{1}{2}\, x''^{\gamma} f_{\mu\beta\gamma} f^{\beta\alpha}\right) X_{\alpha\nu} - \left(\Gamma_{\nu\beta}{}^{\alpha}\, x'^{\beta} + \frac{1}{2}\, x''^{\gamma} f_{\nu\beta\gamma} f^{\beta\alpha}\right) X_{\mu\alpha}\right] \frac{\partial x^{\mu}}{\partial y^i} \frac{\partial x^{\nu}}{\partial y^j},$$

which shows that the quantity in the brackets on the right is a covariant tensor of rank 2.

Evidently the differentiation process θ applies to a tensor of any type and rank and yields a tensor of the same type and rank. The general rule may be formulated as follows:

To find the result of the θ-operation with respect to a curve

$$x^{\alpha} = x^{\alpha}(u) \qquad (\alpha = 1, \cdots, n)$$

when applied to a tensor $X^{\cdots\cdots}_{\cdots\cdots}$ of any type, form the derivative

$$\frac{dX^{\cdots\cdots}_{\cdots\cdots}}{du}$$

and for each contravariant index μ, $X^{\cdot\cdot\mu\cdot}_{\cdots\cdots}$, add

$$\left(\Gamma_{\alpha\beta}{}^{\mu}\, x'^{\beta} + \frac{1}{2}\, x''^{\gamma} f_{\alpha\beta\gamma} f^{\beta\mu}\right) X^{\cdot\cdot\alpha\cdot}_{\cdots\cdots},$$

and for each covariant index μ, $X^{\cdots\cdots}_{\cdots\mu\cdot}$, subtract

$$\left(\Gamma_{\mu\beta}{}^{\alpha}\, x'^{\beta} + \frac{1}{2}\, x''^{\gamma} f_{\mu\beta\gamma} f^{\beta\alpha}\right) X^{\cdots\cdots}_{\cdots\alpha\cdot}.$$

We now consider the application of the differentiation process to a product. Suppose ξ and η are each covariant vectors. Then

$$X_{\mu\nu} = \xi_{\mu}\,\eta_{\nu}$$

is a covariant tensor of rank 2. Consider the expression

$$(\theta\,\xi_{\mu})\,\eta_{\nu} + \xi_{\mu}\,(\theta\,\eta_{\nu}) = \frac{d\xi_{\mu}}{du}\,\eta_{\nu} + \xi_{\mu}\frac{d\eta_{\nu}}{du} - \left(\Gamma_{\mu\beta}{}^{\alpha}\, x'^{\beta} + \frac{1}{2}\, x''^{\gamma} f_{\mu\beta\gamma} f^{\beta\alpha}\right)\xi_{\alpha}\,\eta_{\nu} - \left(\Gamma_{\nu\beta}{}^{\alpha}\, x'^{\beta} + \frac{1}{2}\, x''^{\gamma} f_{\nu\beta\gamma} f^{\beta\alpha}\right)\xi_{\mu}\,\eta_{\alpha}$$
$$= \theta\, X_{\mu\nu}.$$

*Therefore, the distributive rule for the θ-process applied to a product of the form $X_{\mu\nu} = \xi_\mu \eta_\nu$ holds as for ordinary differentiation.**

Consider the mixed tensor $\varphi_\mu^{\ \nu}$. Then

$$\theta\varphi_\mu^{\ \nu} = \frac{d\varphi_\mu^{\ \nu}}{du} - \left(\Gamma_{\mu\beta}^{\ \alpha} x'^\beta + \frac{1}{2} x''^\gamma f_{\mu\beta\gamma} f^{\beta\alpha}\right)\varphi_\alpha^{\ \nu} \tag{25}$$
$$+ \left(\Gamma_{\alpha\beta}^{\ \nu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\nu}\right)\varphi_\mu^{\ \alpha}.$$

If we *contract* this tensor in the usual way by equating the upper and lower indices we obtain a scalar invariant function $S = \varphi_\mu^{\ \mu}$. The expression (25) then becomes

$$\theta S = \theta\varphi_\mu^{\ \mu} = \frac{dS}{du} + \begin{bmatrix} -\left(\Gamma_{\alpha\beta}^{\ \mu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right) \\ +\left(\Gamma_{\alpha\beta}^{\ \mu} x'^\beta + \frac{1}{2} x''^\gamma f_{\alpha\beta\gamma} f^{\beta\mu}\right)\end{bmatrix}\varphi_\mu^{\ \alpha}$$
$$= \frac{dS}{du}.$$

Hence we have the theorem

If S is a scalar function expressed as a contraction of a tensor, $S = \varphi_\mu^{\ \mu}$, then θS is the same as the ordinary derivative of S with respect to u.

Since a product of tensors is a tensor, this theorem is true for any scalar function which is expressed as a contraction of tensors.

In the Riemann geometry the covariant derivative of the fundamental tensor $g_{\alpha\beta}$ is zero, and we shall now show that this property generalizes:

$$\theta f_{\mu\nu} = \frac{df_{\mu\nu}}{du} - \left(\Gamma_{\mu\beta}^{\ \alpha} x'^\beta + \frac{1}{2} x''^\gamma f_{\mu\beta\gamma} f^{\beta\alpha}\right) f_{\alpha\nu}$$
$$- \left(\Gamma_{\nu\beta}^{\ \alpha} x'^\beta + \frac{1}{2} x''^\gamma f_{\nu\beta\gamma} f^{\beta\alpha}\right) f_{\mu\alpha}$$
$$= \frac{\partial f_{\mu\nu}}{\partial x^\beta} x'^\beta - \Gamma_{\mu\beta}^{\ \alpha} x'^\beta f_{\alpha\nu} - \Gamma_{\nu\beta}^{\ \alpha} x'^\beta f_{\mu\alpha}$$
$$+ f_{\mu\nu\gamma} x''^\gamma - \frac{1}{2} x''^\gamma f_{\mu\beta\gamma} f^{\beta\alpha} f_{\alpha\nu} - \frac{1}{2} x''^\gamma f_{\nu\beta\gamma} f^{\beta\alpha} f_{\mu\alpha}.$$

By means of (16) and (7) this may be written

$$\theta f_{\mu\nu} = \left(\frac{\partial f_{\mu\nu}}{\partial x^\beta} - [\mu\beta, \nu] - [\nu\beta, \mu]\right) x'^\beta + \left(f_{\mu\nu\gamma} - \frac{1}{2} f_{\mu\nu\gamma} - \frac{1}{2} f_{\nu\mu\gamma}\right) x''^\gamma$$
$$= 0,$$

by (17).

* This theorem has been established here for two tensors both covariant but it is easily seen to be true for the product of tensors of any type.

That is, the θ-derivative of the covariant tensor $f_{\alpha\beta}$ is zero.

A direction X^α defined along a curve C in a Riemann space remains *parallel* in the Levi-Civita sense* if it satisfies the system of linear differential equations obtained by equating to zero the expression (13). The θ-process here developed is clearly a generalization of (13) since it reduces to the latter if $f_{\alpha\beta}$ are merely point functions, i. e., in case the space is Riemannian. A theorem by Levi-Civita† generalizes at once.

Let ξ_1 and ξ_2 be two contravariant vectors which are defined at each point of the curve C as functions of x and its derivatives with respect to u. Moreover, suppose ξ_1 and ξ_2 satisfy the system of linear differential equations

$$\theta\xi^\alpha = 0 \qquad (\alpha = 1, \cdots, n) \tag{26}$$

which are associated with the curve. Then if the vectors are measured with respect to the tangent direction of the curve C, the norms of ξ_1 and ξ_2 remain constant, and the angle between ξ_1 and ξ_2 remains constant as the vectors move along the curve.

For each of the expressions

$$f_{\alpha\beta}\,\xi_1^\alpha\,\xi_1^\beta, \qquad f_{\alpha\beta}\,\xi_2^\alpha\,\xi_2^\beta, \qquad f_{\alpha\beta}\,\xi_1^\alpha\,\xi_2^\beta$$

is a scalar function formed by a contraction of tensors, and therefore, considering the last one,

$$\begin{aligned} \frac{d}{du}(f_{\alpha\beta}\,\xi_1^\alpha\,\xi_2^\beta) &= \theta(f_{\alpha\beta}\,\xi_1^\alpha\,\xi_2^\beta) \\ &= (\theta f_{\alpha\beta})\,\xi_1^\alpha\,\xi_2^\beta + f_{\alpha\beta}(\theta\xi_1^\alpha)\,\xi_2^\beta + f_{\alpha\beta}\,\xi_1^\alpha(\theta\xi_2^\beta) \\ &= 0, \end{aligned}$$

since $\theta f_{\alpha\beta} = 0$, and ξ_1 and ξ_2 satisfy (26). Hence, each of the three above expressions is a constant, which establishes the theorem.

Let $\xi_1, \xi_2, \cdots, \xi_n$ be a fundamental set of solutions of the system (26), and suppose $\eta_1, \eta_2, \cdots, \eta_n$ to be a linearly independent set of vectors expressed linearly in terms of $\xi_1, \cdots, \xi_n$ and with the additional property that $\eta_1, \cdots, \eta_n$ constitute a unitary orthogonal set.‡ Then as an immediate consequence of the theorem just given it follows that *the set $\eta_1, \cdots, \eta_n$ being initially a unitary orthogonal one remains unitary orthogonal all along the curve.*

* T. Levi-Civita, loc. cit.

† T. Levi-Civita, loc. cit., p. 182.

‡ A method of determining a unitary orthogonal system is discussed in the next section.

4. Orthogonalization process. Frenet formulas.* Let the equations of a curve C be

$$x^\alpha = x^\alpha(t), \qquad t_1 \leqq t \leqq t_2 \qquad (\alpha = 1, \cdots, n),$$

where now the curve is referred to the arc length t in the sense of (3), as the parameter. Such a selection of the parameter is always possible. For, if the equations of the curve are given in terms of an arbitrary parameter u it follows from the definition of the arc length

$$t = \int_{u_1}^{u} F\left(x, \frac{dx}{du}\right) du$$

that

$$F\left(x, \frac{dx}{dt}\right) = 1$$

is a necessary and sufficient condition that the independent variable be the arc length. Clearly this condition can always be satisfied by virtue of the homogeneous property of F. We shall assume for the remainder of this paper that such a choice of the independent variable has been made, and hereafter the primes will denote derivatives with respect to the arc length t.

As a consequence of the choice of the arc length as parameter it follows that the tangent vector $x' = dx/dt$ is a unitary vector, for by (5)

$$f_{\alpha\beta}(x, x')x'^\alpha x'^\beta = F^2(x, x').$$

The tangent vector x', which we denote hereafter by ξ_1, is a contravariant tensor of rank 1. Then $\xi_2 = \theta\xi_1$ is also a contravariant vector; likewise $\xi_3 = \theta\xi_2, \cdots$, where the θ-operation is defined by (23) with arc length as the independent variable. Hence we can associate with each point of the curve a system of contravariant vectors, $\xi_1, \xi_2, \cdots, \xi_n$ which are obtained sequentially by repeated application of the directional derivative θ, thus

$$\xi_1 = x', \qquad \xi_k = \theta\xi_{k-1} \qquad (k = 2, 3, \cdots, n).$$

We suppose this system of vectors to be linearly independent so that they will form a basis for the whole vector space at the point of the curve C at which they are taken. Our first problem is to replace $\xi_1, \cdots, \xi_p$ $(p = 1, \cdots, n)$ by a set of vectors $\eta_1, \cdots, \eta_p$ which are equivalent in the

* This section follows very closely the paper by W. Blaschke, loc. cit.

sense that they define the same space, and which shall have the additional property that they constitute a *unitary orthogonal* system, which is defined by

$$(\eta_i, \eta_k)_{x'} = f_{\alpha\beta}(x, x')\, \eta_i^\alpha \eta_k^\beta = \delta_{ik},$$

where $\delta_{ii} = 1$ and $\delta_{ik} = 0$ for $i \neq k$. It is essential for use in the later development that this norming and orthogonalizing shall be with respect to the tangent direction x' as is indicated by the notation.

For simplicity of notation we denote the inner product, $(\xi_p, \xi_q)_{x'}$, by (p, q). We now define a set of vectors by the equations

$$\zeta_1 = \xi_1,\ \zeta_p = \begin{vmatrix} (1,1) & \cdots & (1, p-1) & \xi_1 \\ \cdots & \cdots & \cdots & \cdots \\ (p,1) & \cdots & (p, p-1) & \xi_p \end{vmatrix} \qquad (p = 2, \cdots, n).$$

Then $f_{\alpha\beta}\, \zeta_p^\alpha \zeta_q^\beta = 0$ for $q = 1, 2, \cdots, p-1$, i. e., for $p > q$. The vectors $\zeta_1, \cdots, \zeta_n$ then constitute an orthogonal system; to make them into a unitary system it will only be necessary to divide each vector by the square root of its norm. Now

$$f_{\alpha\beta}\, \zeta_p^\alpha \zeta_p^\beta = D_{p-1}\,(f_{\alpha\beta}\, \xi_p^\alpha \zeta_p^\beta) = D_{p-1}\, D_p,$$

where D_p is defined by

$$(27) \qquad D_0 = 1,\ D_p = \begin{vmatrix} (1,1) & \cdots & (1,p) \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ (p,1) & \cdots & (p,p) \end{vmatrix}, \qquad (p = 1, \cdots, n).$$

Since the norm of $\zeta_p > 0$, it follows that $D_p > 0$. Hence the system of vectors $\eta_1, \cdots, \eta_n$ given by

$$(28)\ \eta_1 = \xi_1,\ \eta_p = \frac{1}{\sqrt{D_{p-1}\, D_p}} \begin{vmatrix} (1,1) & \cdots & (1, p-1) & \xi_1 \\ \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ (p,1) & \cdots & (p, p-1) & \xi_p \end{vmatrix} \quad (p = 2, \cdots, n)$$

is the unitary orthogonal system desired. Clearly they are linearly independent and form a basis for the whole vector space at the point of the curve C at which they are taken.

The vector η_p being a linear combination, with scalar coefficients, of contravariant vectors is itself a contravariant vector. Then $\theta\,\eta_p$ is a contravariant vector and can therefore be expressed as a linear combination of $\eta_1, \cdots, \eta_n$ in the form

$$(29) \qquad \theta\,\eta_p^\alpha = C_{pq}\, \eta_q^\alpha \qquad (q \text{ summed from 1 to } n),$$

where

$$C_{pq} = f_{\alpha\beta}(\theta\eta_p^\alpha)\eta_q^\beta,$$

the coefficients C_{pq} being scalars or *invariants*. Following Blaschke and others we call these invariants the *curvatures* of the curve.

From

$$f_{\alpha\beta}\eta_p^\alpha\eta_q^\beta = 0, \qquad p \neq q,$$

we obtain

$$\theta(f_{\alpha\beta}\eta_p^\alpha\eta_q^\beta) = f_{\alpha\beta}(\theta\eta_p^\alpha)\eta_q^\beta + f_{\alpha\beta}\eta_p^\alpha(\theta\eta_q^\beta) = 0,$$

where we have made use of the distributive law for the θ-process applied to a product, the fact that $\theta f_{\alpha\beta} = 0$, and the theorem that the θ-derivative for a scalar function expressed as a contraction of tensors yields the same result as the ordinary derivative. Hence

$$C_{pq} + C_{qp} = 0. \tag{30}$$

Now η_p is a linear combination of $\xi_1, \cdots, \xi_p$, from which it follows that $\theta\eta_p$ is a linear combination of $\xi_1, \cdots, \xi_{p+1}$ or of $\eta_1, \cdots, \eta_{p+1}$ only, and therefore

$$C_{pq} = 0 \text{ for } q > p + 1. \tag{31}$$

As a consequence of (30) and (31) we see that the matrix of coefficients C_{pq} is of the form

$$\| C_{pq} \| = \left\| \begin{array}{ccccc} 0 & +\dfrac{1}{\varrho_1} & 0 & \cdots & 0 \\ -\dfrac{1}{\varrho_1} & 0 & +\dfrac{1}{\varrho_2} & \cdots & 0 \\ 0 & -\dfrac{1}{\varrho_2} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right\|$$

where

$$\frac{1}{\varrho_p} = f_{\alpha\beta}(\theta\eta_p^\alpha)\eta_{p+1}^\beta, \qquad 0 < p < n,$$

$$\frac{1}{\varrho_0} = \frac{1}{\varrho_n} = 0.$$

We now write (29) in the form

$$\theta\eta_p = -\frac{1}{\varrho_{p-1}}\eta_{p-1} + \frac{1}{\varrho_p}\eta_{p+1}, \qquad 0 < p < n, \tag{32}$$

$$\frac{1}{\varrho_0} = \frac{1}{\varrho_n} = 0.$$

These formulas are the analogues of the well known Frenet formulas associated with a twisted curve in space.

It is desirable to compute the curvatures $1/\varrho_p$ in terms of the θ-derivatives along the curve. Notice that

$$\begin{aligned} \theta(p, q) &= \theta(f_{\alpha\beta}\, \xi_p^\alpha\, \xi_q^\beta) \\ &= (\theta f_{\alpha\beta})\, \xi_p^\alpha\, \xi_q^\beta + f_{\alpha\beta}\,(\theta\, \xi_p^\alpha)\, \xi_q^\beta + f_{\alpha\beta}\, \xi_p^\alpha\,(\theta\, \xi_q^\beta) \\ &= (p+1, q) + (p, q+1) \end{aligned}$$

since $\theta f_{\alpha\beta} = 0$. From (28)

$$\begin{aligned} \theta\, \eta_p^\alpha = {} & \left[\theta\left(\frac{1}{\sqrt{D_{p-1}\, D_p}}\right)\right] \sqrt{D_{p-1}\, D_p}\; \eta_p^\alpha \\ & + \frac{1}{\sqrt{D_{p-1}\, D_p}} \left[\begin{vmatrix} (2,1) & (1,2) & \cdots & \xi_1^\alpha \\ (3,1) & (2,2) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (p+1,1) & \cdots & \cdots & \xi_p^\alpha \end{vmatrix} + \begin{vmatrix} (1,1) & (2,2) & \cdots & \xi_1^\alpha \\ (1,2) & (3,2) & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (p,1) & (p+1,2) & \cdots & \xi_p^\alpha \end{vmatrix} \right. \\ & \left. + \cdots + \begin{vmatrix} (1,1) & \cdots & (1,p-1) & \xi_2^\alpha \\ \cdots & \cdots & \cdots & \cdots \\ (p,1) & \cdots & (p,p-1) & \xi_{p+1}^\alpha \end{vmatrix} \right]. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{\varrho_p} &= f_{\alpha\beta}\,(\theta\, \eta_p^\alpha)\, \eta_{p+1}^\beta \\ &= \frac{1}{\sqrt{D_{p-1}\, D_p}} \begin{vmatrix} (1,1) & \cdots & (1,p-1) & 0 \\ \cdots & \cdots & \cdots & \cdots \\ (p-1,1) & \cdots & (p-1,p-1) & 0 \\ (p,1) & \cdots & (p,p-1) & (\xi_{p+1}, \eta_{p+1}) \end{vmatrix} \\ &= \frac{1}{\sqrt{D_{p-1}\, D_p}} \cdot D_{p-1} \cdot \frac{1}{\sqrt{D_p\, D_{p+1}}} \cdot D_{p+1}, \end{aligned}$$

and, therefore,

$$\frac{1}{\varrho_p} = \frac{\sqrt{D_{p-1}\, D_{p+1}}}{D_p}. \tag{33}$$

This theorem may be formulated as follows:

Let $x^\alpha = x^\alpha(t)$, $t_1 \leqq t \leqq t_2$ $(\alpha = 1, \cdots, n)$ be the equations of a curve referred to the arc length as parameter. Form the system of contravariant *vectors* $\xi_1, \cdots, \xi_n$ *defined by* $\xi_1 = x'$, $\xi_k = \theta\,\xi_{k-1}$ $(k = 2, \cdots, n)$. *Let these vectors be normed and orthogonalized with respect to the tangent direction to the curve yielding a unitary orthogonal set* $\eta_1, \cdots, \eta_n$ *of principal directions associated with each point of the curve. Then the Frenet formulas may be written in the form*

$$\theta\,\eta_p = -\frac{1}{\varrho_{p-1}}\eta_{p-1} + \frac{1}{\varrho_p}\eta_{p+1}, \qquad 0 < p < n,$$

$$\frac{1}{\varrho_0} = \frac{1}{\varrho_n} = 0,$$

where the pth curvature $1/\varrho_p$ *is given by* (33).

NOTE: I regret to say that adequate reference has not here been made to the paper, *A generalization of the Riemannian line-element* by J. L. Synge, these Transactions, this volume, pp. 61–67. When I returned the proof sheets of my paper, I was aware that Mr. Synge had written a paper on the same general subject, but the scanty and indirect account I had of his paper did not indicate much overlapping. It was only upon publication of Mr. Synge's paper that an adequate account of his results became available to me.

The same remarks apply to § 4 of my paper *Reduction of Euler's equations to a canonical form,* Bulletin of the American Mathematical Society, vol. 31 (1925).

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